Exercises for Lecture Course on Optimal Control and Estimation (OCE)
Albert-Ludwigs-Universität Freiburg – Summer Term 2014

Exercise 1 - Discrete and Continuous Dynamic Systems (deadline: 12.5.2014)

Prof. Dr. Moritz Diehl and Greg Horn

Aim of this exercise is to get used to dealing with dynamic systems. We will start with a discrete time popoluation model. Then we will do a continuous-time nonlinear differential equation (a 2-dimensional aircraft model) and transform it into discrete time by numerical simulation.

1 Population Model

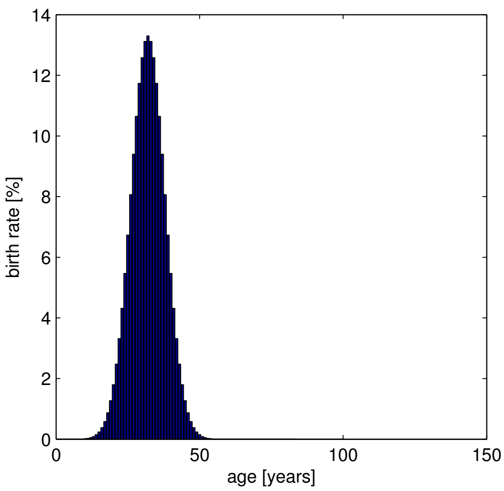
Consider a linear model of a population. State vector $x \in \mathbb{R}^{100}$ represents the population of each age group. Let $x_i(k)$ mean the number of people of age i during year k. For instance, $x_6(2014)$ would be the number of people who are 6 years old in year 2014. Each year babies (0-year-olds) are formed depending on a linear birth rate:

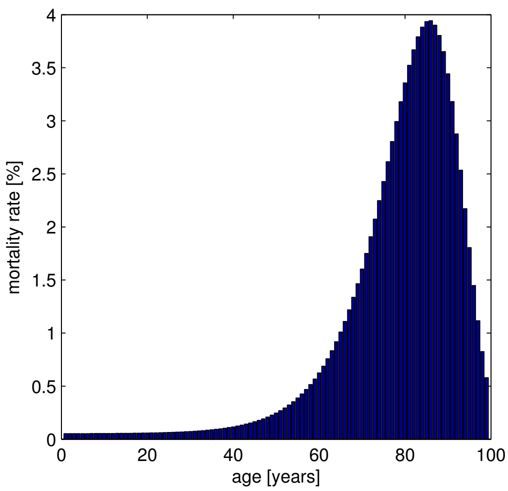
$$x_0(k) = \sum_{j=0}^{99} \beta_j x_j(k) \tag{1}$$

Each year most of the population ages by one year, except for a fraction who die according to mortality rate μ:

$$x_{i+1}(k+1) = x_i(k) - \mu_i x_i(k) \qquad i = 1, \ldots, 98 \tag{2}$$

β and μ will be provided for you at
`NUMOPCO/OCE-COURSE/exercises/exercise1/code/birth_mortality_rates.m`
and look like:





Tasks

1. Write the discrete time model in the form of

$$x(k+1) = A\,x(k) \tag{3}$$

2. Lord of the Flies: Setting an initial population of 100 four-year-olds, and no other people, simulate the system for 150 years. Make a 3-d plot of the population, with axes {year, age, population}.

3. Eigen decomposition: Plot the eigenvalues of A in the complex plane. Plot the real part of the two eigenvectors of A which have largest eigenvalue magnitude

   Is this system stable? What is the significance of these eigenvectors with large eigenvalues?

4. Run two simulations, each time using an eigenvector from the previous question as $x(0)$. What is the significance of this result?

2 Paper Airplane Modeling

Consider a two-dimensional model of an airplane with states $x = [p_x, p_z, v_x, v_z]$ where position $\vec{p} = [p_x, p_z]$ and velocity $\vec{v} = [v_x, v_z]$ are vectors in the $x - z$ directions. Since the TA for this class has an aerospace background, we will use the standard aerospace convention that $\hat{x}$ is forward and $\hat{z}$ is DOWN, so altitude is $-p_z$. The system has one control $u = [\alpha]$, where α is the aerodynamic angle of attack in radians. The system dynamics are:

$$\frac{d}{dt}\begin{pmatrix} p_x \\ p_z \\ v_x \\ v_z \end{pmatrix} = \begin{pmatrix} v_x \\ v_z \\ F_x/m \\ F_z/m \end{pmatrix} \tag{4}$$

where $m = 2.0$ is the mass of the airplane. The forces $\vec{F}$ on the airplane are

$$\vec{F} = \vec{F}_{\text{lift}} + \vec{F}_{\text{drag}} + \vec{F}_{\text{gravity}} \tag{5}$$

Lift force $\vec{F}_{\text{lift}}$ is

$$\vec{F}_{\text{lift}} = \frac{1}{2}\rho\|\vec{v}\|^2 C_L(\alpha) S_{\text{ref}}\, \hat{e}_L \tag{6}$$

where lift direction $\hat{e}_L = [v_z, -v_x]/\|\vec{v}\|$, and lift coefficient $C_L = 2\pi\alpha\frac{10}{12}$. S_{ref} is the wing aerodynamic reference area. The drag force $\vec{F}_{\text{drag}}$ is

$$\vec{F}_{\text{drag}} = \frac{1}{2}\rho\|\vec{v}\|^2 C_D(\alpha) S_{\text{ref}}\, \hat{e}_D \tag{7}$$

Drag direction $\hat{e}_D = -\vec{v}/\|\vec{v}\|$, and drag coefficient $C_D = 0.01 + \frac{C_L^2}{10\pi}$. The gravitational force is

$$\vec{F}_{\text{gravity}} = [0, m\,g] \tag{8}$$

Use $\rho = 1.2$, $g = 9.81$, $S_{\text{ref}} = 0.5$.

Tasks

1. Write the continuous time model in the form of

$$\frac{d}{dt}x = f(x, u) \tag{9}$$

2. Simulate the system for 10 seconds using the `ode45` MATLAB function. Plot p_x, p_z, v_x, v_z vs. time, and p_x vs. altitude.

3. Convert the system to the discrete time form

$$x(k+1) = f_d(x(k), u(k)) \tag{10}$$

   using a forward Euler integrator. Simulate this system and compare to `ode45`. Estimating the accuracy by eye, how small do you have to make the time step so that results are similar accuracy to `ode45`? Using the MATLAB functions `tic` and `toc`, how much time does `ode45` take compared to forward Euler for similar accuracy?

4. Re-do the previous item using 4th order Runge-Kutta (RK4) instead of forward Euler. Which is faster (for similar accuracy) among the three methods?

5. **Bonus question:** Linearize the discrete time RK4 system to make an approximate system

$$x(k+1) \approx A\,x(k) + B\,u(k) \tag{11}$$

   using any linearization method you want.

   Plot the Eigenvalues of A in the complex plane. Is the system stable? Is this a problem?