Exercises for Lecture Course on Modelling and System Identification (MSI)
Albert-Ludwigs-Universität Freiburg – Winter Term 2014

Exercise 3: Another review of optimization, systems, and statistics

(to be returned on Nov 11, 2014, 8:15 in HS 101.00.026, or before in building 102, 1st floor, 'Anbau')

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Solutions

1. The gradient of a scalar function $f : \mathbb{R}^n \to \mathbb{R}$ is a vector with the partial derivatives with respect to $x_k, k = 1 \dots n$:

$$\begin{aligned}(\nabla f(x))_k &= \frac{\partial}{\partial x_k}\left[x^\top Q x + c^\top x\right], \\ &= \frac{\partial}{\partial x_k}\left[\sum_{i=1}^n x_i \cdot \sum_{j=1}^n Q_{i,j} x_j + \sum_{i=1}^n c_i x_i\right].\end{aligned}$$

Using the product rule of differentiation, we have

$$= \sum_{j=1}^n Q_{k,j} x_j + \sum_{i=1}^n x_i Q_{i,k} + c_k.$$

Collecting this partial derivatives for $k = 1 \dots n$ in a vector, we get

$$\nabla f(x) = (Q + Q^\top)x + c.$$

The Hessian is the matrix of second partial derivatives.

$$\begin{aligned}(\nabla^2 f(x))_{k,l} &= \frac{\partial}{\partial x_l\, \partial x_k}\left[x^\top Q x + c^\top x\right], \\ &= Q_{k,l} + Q_{l,k},\end{aligned}$$

Or, in matrix form

$$\nabla^2 f(x) = Q + Q^\top.$$

Note that the Hessian is symmetric. This holds generally under continuity of f.

If Q is symmetric and positive definite, it can be inverted. The stationary point x^* amounts to

$$\begin{aligned}0 &= (Q + Q^\top)x^* + c, \\ x^* &= -\frac{1}{2}Q^{-1}c.\end{aligned}$$

This stationary point is a minimizer because of positive definiteness of Q, which implies convexity of f. The minimum function value is then

$$\begin{aligned}f(x^*) &= x^{*\top} Q x^* + c^\top x^*, \\ &= +\frac{1}{4}c^\top Q^{-1} Q Q^{-1} c - \frac{1}{2}c^\top Q^{-1} c, \\ &= -\frac{1}{4}c^\top Q^{-1} c.\end{aligned}$$

2. We use the following facts from basic mechanics, where we denote the position, the velocity, acceleration, force and mass with s, v, a, F, m respectively:

$$
\begin{aligned}
F(t) &= m \cdot a(t), \\
\frac{\mathrm{d}s(t)}{\mathrm{d}t} &= v(t), \\
\frac{\mathrm{d}v(t)}{\mathrm{d}t} &= a(t).
\end{aligned}
$$

For our hockey puck system, we are interested in the 2-D position. We can derive the following equations of motion from the basic facts above:

$$
\begin{aligned}
\frac{\mathrm{d}s_X(t)}{\mathrm{d}t} &= v_X(t), \\
\frac{\mathrm{d}v_X(t)}{\mathrm{d}t} &= F_X(t)/m, \\
\frac{\mathrm{d}s_Y(t)}{\mathrm{d}t} &= v_Y(t), \\
\frac{\mathrm{d}v_Y(t)}{\mathrm{d}t} &= F_Y(t)/m.
\end{aligned}
$$

In state-space form ($\dot{x} = Ax + Bu$) this becomes

$$
\begin{bmatrix} \dot{s_X} \\ \dot{v_X} \\ \dot{s_Y} \\ \dot{v_Y} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} s_X \\ v_X \\ s_Y \\ v_Y \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1/m & 0 \\ 0 & 0 \\ 0 & 1/m \end{bmatrix} \begin{bmatrix} F_X \\ F_Y \end{bmatrix}.
$$