Exercises for Lecture Course on Modelling and System Identification (MSI)
Albert-Ludwigs-Universität Freiburg – Winter Term 2014

Exercise 4: Covariance Estimation in a Single Experiment

(to be returned on Nov 18, 2014, 8:15 in HS 26, or before in building 102, 1st floor, 'Anbau')

Prof. Dr. Moritz Diehl and Robin Verschueren

Solution

2. **Maximum-Likelihood Estimation:**

The likelihood function for just one detector located at $x = 0.5\,\mathrm{mm}$ would be

$$\mathcal{L}_1(\theta_1,\theta_2,\theta_3) = \frac{(\theta_2)^{y(1)} \exp\left(-\frac{(0.5-\theta_1)^2}{\theta_3}\cdot y(1)\right)\cdot \exp\left(-\theta_2 \exp(-\frac{(0.5-\theta_1)^2}{\theta_3})\right)}{y(1)!}.$$

The negative log-likelihood function would then be

$$\begin{aligned}-\log(\mathcal{L}_1) &= -\left(-\log(y(1)!) + y(1)\log(\theta_2) - y(1)\frac{(0.5-\theta_1)^2}{\theta_3} - \theta_2\exp(-\frac{(0.5-\theta_1)^2}{\theta_3})\right)\\ &= C_1 + y(1)\left(\frac{(0.5-\theta_1)^2}{\theta_3} - \log(\theta_2)\right) + \theta_2\exp(-\frac{(0.5-\theta_1)^2}{\theta_3}).\end{aligned}$$

For N detectors, the likelihood function amounts to

$$\begin{aligned}\mathcal{L}_N(\theta_1,\theta_2,\theta_3) = &\frac{(\theta_2)^{y(1)+\ldots+y(N)}\exp\left(-\frac{(0.5-\theta_1)^2}{\theta_3}\cdot y(1) - \ldots - \frac{(N-0.5-\theta_1)^2}{\theta_3}\cdot y(N)\right)\cdot\ldots}{y(1)!\cdot y(2)!\cdot\ldots}\\ &\frac{\ldots\cdot\exp\left(-\theta_2\exp(-\frac{(0.5-\theta_1)^2}{\theta_3}) - \ldots - \theta_2\exp(-\frac{(N-0.5-\theta_1)^2}{\theta_3})\right)}{\ldots\cdot y(N)!},\\ = &\frac{(\theta_2)^{\sum_{k=1}^N y(k)}\exp\left(-\frac{\sum_{k=1}^N y(k)\cdot(k-0.5-\theta_1)^2}{\theta_3}\right)\exp\left(-\theta_2\ \sum_{k=1}^N \exp(-\frac{(k-0.5-\theta_1)^2}{\theta_3})\right)}{y(1)!\cdot y(2)!\cdot\ldots\cdot y(N)!}.\end{aligned}$$

The negative log-likelihood is then equal to

$$-\log(\mathcal{L}_N) = C_2 - C_3\log(\theta_2) + \frac{\sum_{k=1}^N y(k)\cdot(k-0.5-\theta_1)^2}{\theta_3} + \theta_2\sum_{k=1}^N \exp\left(-\frac{(k-0.5-\theta_1)^2}{\theta_3}\right).$$