Optimal Control of Airborne Wind Energy Systems

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Outline

1 Control of Tethered Airfoils

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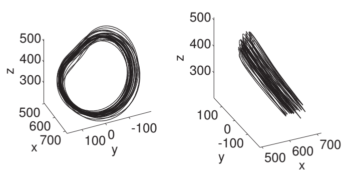
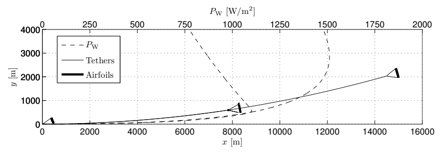
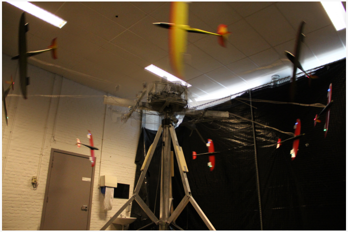
Control of Tethered Airfoils

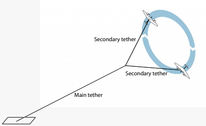


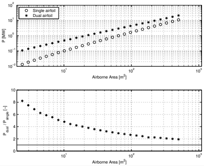


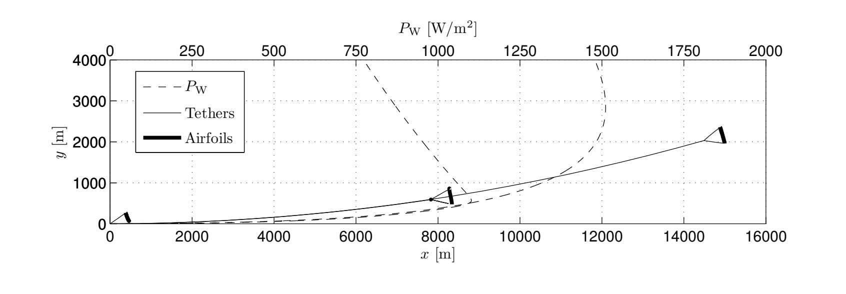


Dual Airfoils [Zanon et al. 2013]

- More advantageous than single airfoil
- More complex, nonlinear and unstable







Dual Airfoils

Pointmass model in spherical coordinates [Williams et al., 2008]

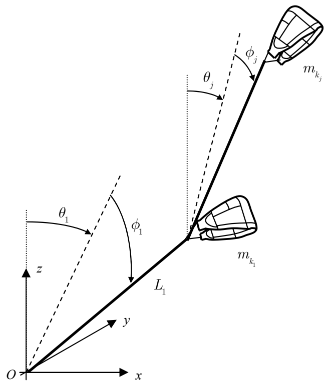


Fig. 2. Multiple kite model with kites on single line.

Spherical Coordinates

Dual airfoil model in spherical coordinates...

$$
\boldsymbol{A} = \begin{bmatrix}
A_{1,1} & 0 & 0 & A_{1,4} & A_{1,5} & 0 & A_{1,7} & A_{1,8} & 0 \\
0 & A_{2,2} & 0 & A_{2,4} & A_{2,5} & 0 & A_{2,7} & A_{2,8} & 0 \\
0 & 0 & 1 & A_{3,4} & A_{3,5} & 0 & A_{3,7} & A_{3,8} & 0 \\
A_{1,4} & A_{2,4} & 0 & A_{4,4} & 0 & 0 & 0 & 0 & 0 \\
A_{1,5} & A_{2,5} & 0 & 0 & A_{5,5} & 0 & 0 & 0 & 0 \\
A_{6,1} & A_{6,2} & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
A_{1,7} & A_{2,7} & 0 & 0 & 0 & 0 & A_{7,7} & 0 & 0 \\
A_{1,8} & A_{2,8} & 0 & 0 & 0 & 0 & 0 & A_{8,8} & 0 \\
A_{9,1} & A_{9,2} & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix} \tag{35}
$$

where

$$A_{1,1} = 1/3\,L_0^2 \cos^2\phi_0 \left(\rho_0 L_0 + 3m_2 + 3L_1\rho_1 + 3L_2\rho_2 + 3m_1\right) \tag{36}$$

$$A_{1,4} = 1/2\,L_0 L_1 \cos\phi_1 \cos\phi_0 \left(\sin\theta_1 \sin\theta_0 + \cos\theta_1 \cos\theta_0\right)\left(L_1\rho_1 + 2m_1\right) \tag{37}$$

$$A_{1,5} = -1/2\,L_0 L_1 \sin\phi_1 \cos\phi_0 \left(\sin\theta_1 \cos\theta_0 - \cos\theta_1 \sin\theta_0\right)\left(L_1\rho_1 + 2m_1\right) \tag{38}$$

$$A_{1,7} = 1/2\,L_0 L_2 \cos\phi_2 \cos\phi_0 \left(\sin\theta_2 \sin\theta_0 + \cos\theta_0 \cos\theta_2\right)\left(L_2\rho_2 + 2m_2\right) \tag{39}$$

$$A_{1,8} = -1/2\,L_0 L_2 \sin\phi_2 \cos\phi_0 \left(\sin\theta_2 \cos\theta_0 - \sin\theta_0 \cos\theta_2\right)\left(L_2\rho_2 + 2m_2\right) \tag{40}$$

$$A_{2,2} = 1/3\,L_0^2 \left(\rho_0 L_0 + 3m_2 + 3L_1\rho_1 + 3L_2\rho_2 + 3m_1\right) \tag{41}$$

$$A_{2,4} = 1/2\,L_0 L_1 \cos\phi_1 \sin\phi_0 \left(\sin\theta_1 \cos\theta_0 - \cos\theta_1 \sin\theta_0\right)\left(L_1\rho_1 + 2m_1\right) \tag{42}$$

$$A_{2,5} = 1/2\,L_0 L_1 \left(\sin\phi_1 \sin\theta_1 \sin\phi_0 \sin\theta_0 + \cos\phi_1 \cos\phi_0 + \sin\phi_1 \cos\theta_1 \sin\phi_0 \cos\theta_0\right)\left(L_1\rho_1 + 2m_1\right) \tag{43}$$

$$A_{2,7} = 1/2\,L_0 L_2 \sin\phi_0 \cos\phi_2 \left(\sin\theta_2 \cos\theta_0 - \sin\theta_0 \cos\theta_2\right)\left(L_2\rho_2 + 2m_2\right) \tag{44}$$

$$A_{2,8} = 1/2\,L_0 L_2 \left(\sin\phi_2 \cos\theta_2 \sin\phi_0 \cos\theta_0 + \sin\phi_2 \sin\theta_2 \sin\phi_0 \sin\theta_0 + \cos\phi_2 \cos\phi_0\right)\left(L_2\rho_2 + 2m_2\right) \tag{45}$$

$$A_{3,4} = 1/2\,L_1 \cos\phi_1 \cos\phi_0 \left(\cos\theta_1 \sin\theta_0 - \cos\theta_0 \sin\theta_1\right)\left(L_1\rho_1 + 2m_1\right) \tag{46}$$

$$A_{3,5} = -1/2\,L_1 \left(\sin\phi_1 \sin\theta_1 \cos\phi_0 \sin\theta_0 - \cos\phi_1 \sin\phi_0 + \sin\phi_1 \cos\theta_1 \cos\phi_0 \cos\theta_0\right)\left(L_1\rho_1 + 2m_1\right) \tag{47}$$

$$A_{3,7} = -1/2\,L_2 \cos\phi_0 \cos\phi_2 \left(\sin\theta_2 \cos\theta_0 - \cos\theta_2 \sin\theta_0\right)\left(L_2\rho_2 + 2m_2\right) \tag{48}$$

$$A_{3,8} = 1/2\,L_2 \left(-\sin\phi_2 \sin\theta_2 \cos\phi_0 \sin\theta_0 + \cos\phi_2 \sin\phi_0 - \sin\phi_2 \cos\theta_2 \cos\phi_0 \cos\theta_0\right)\left(L_2\rho_2 + 2m_2\right) \tag{49}$$

$$A_{4,4} = 1/3\,L_1^2 \cos^2\phi_1 \left(L_1\rho_1 + 3m_1\right) \tag{50}$$

$$A_{5,5} = 1/3\,L_1^2 \left(L_1\rho_1 + 3m_1\right) \tag{51}$$

$$A_{6,1} = -L_0 \cos\phi_1 \cos\phi_0 \left(\cos\theta_1 \sin\theta_0 - \cos\theta_0 \sin\theta_1\right)\left(L_1\rho_1 + m_1\right) \tag{52}$$

$$A_{6,2} = L_0 \left(\sin\phi_1 \cos\phi_0 - \cos\phi_1 \sin\theta_1 \sin\phi_0 \sin\theta_0 - \cos\phi_1 \cos\theta_1 \sin\phi_0 \cos\theta_0\right)\left(L_1\rho_1 + m_1\right) \tag{53}$$

$$A_{7,7} = 1/3\,L_2^2 \cos^2\phi_2 \left(L_2\rho_2 + 3m_2\right) \tag{54}$$

$$A_{8,8} = 1/3\,L_2^2 \left(L_2\rho_2 + 3m_2\right) \tag{55}$$

$$A_{9,1} = L_0 \cos\phi_2 \cos\phi_0 \left(\sin\theta_2 \cos\theta_0 - \cos\theta_2 \sin\theta_0\right)\left(L_2\rho_2 + m_2\right) \tag{56}$$

$$A_{9,2} = L_0 \left(\sin\phi_2 \cos\phi_0 - \cos\phi_2 \sin\theta_2 \sin\phi_0 \sin\theta_0 - \cos\phi_2 \cos\theta_2 \sin\phi_0 \cos\theta_0\right)\left(L_2\rho_2 + m_2\right) \tag{57}$$

The right-hand side of the equations of motion are given by

$$
\begin{aligned}
b_1 = {} & 2m_2 L_2 \sin(\phi_2)\dot\phi_2 \cos(\theta_2)\dot\theta_2 L_0 \cos(\phi_0)\cos(\theta_0) - 2\rho_1 L_0 \left(\cos(\phi_0)\right)^2 \dot\theta_0 L_1 \dot L_0 \\
& -\rho_1 \dot L_1^2 \cos(\phi_1)\sin(\theta_1) L_0 \cos(\phi_0)\cos(\theta_0) - 2\rho_1 \dot L_1 \cos(\phi_1)\sin(\theta_1)\dot\theta_1 L_0 \cos(\phi_0)\sin(\theta_0) L_1 \\
& +\rho_1 \dot L_1^2 \cos(\phi_1)\cos(\theta_1) L_0 \cos(\phi_0)\sin(\theta_0) + \rho_1 L_1^2 \sin(\phi_1)\dot\phi_1 \sin(\theta_1)\dot\theta_1 L_0 \cos(\phi_0)\sin(\theta_0) \\
& +\rho_1 L_1^2 \sin(\phi_1)\dot\phi_1 \cos(\theta_1)\dot\theta_1 L_0 \cos(\phi_0)\cos(\theta_0) - 1/2\rho_1 L_1^2 \cos(\phi_1)\dot\phi_1^2 \cos(\theta_1) L_0 \cos(\phi_0)\sin(\theta_0) \\
& +2\rho_2 \dot L_2 \sin(\phi_2)\dot\phi_2 \sin(\theta_2) L_0 \cos(\phi_0)\cos(\theta_0) L_2 + 2\rho_2 L_0^2 \cos(\phi_0)\dot\theta_0 L_2 \sin(\phi_0)\dot\phi_0 \\
& -2\rho_2 \dot L_2 \cos(\phi_2)\cos(\theta_2)\dot\theta_2 L_0 \cos(\phi_0)\cos(\theta_0) L_2 - \rho_2 \dot L_2^2 \cos(\phi_2)\sin(\theta_2) L_0 \cos(\phi_0)\cos(\theta_0) \\
& -2\rho_2 \dot L_2 \sin(\phi_2)\dot\phi_2 \cos(\theta_2) L_0 \cos(\phi_0)\sin(\theta_0) L_2 - 2\rho_2 \dot L_2 \cos(\phi_2)\sin(\theta_2)\dot\theta_2 L_0 \cos(\phi_0)\sin(\theta_0) L_2 \\
& +\rho_2 \dot L_2^2 \cos(\phi_2)\cos(\theta_2) L_0 \cos(\phi_0)\sin(\theta_0) + \rho_2 L_2^2 \sin(\phi_2)\dot\phi_2 \cos(\theta_2)\dot\theta_2 L_0 \cos(\phi_0)\cos(\theta_0) \\
& +\rho_2 L_2^2 \sin(\phi_2)\dot\phi_2 \sin(\theta_2)\dot\theta_2 L_0 \cos(\phi_0)\sin(\theta_0) + 2\rho_1 L_0^2 \cos(\phi_0)\dot\theta_0 L_1 \sin(\phi_0)\dot\phi_0 \\
& -1/2\rho_2 L_2^2 \cos(\phi_2)\cos(\theta_2)\dot\theta_2^2 L_0 \cos(\phi_0)\sin(\theta_0) + 2m_1 L_1 \sin(\phi_1)\dot\phi_1 \sin(\theta_1)\dot\theta_1 L_0 \cos(\phi_0)\sin(\theta_0) \\
& -2m_1 L_0 \left(\cos(\phi_0)\right)^2 \dot\theta_0 \dot L_0 + m_1 L_1 \cos(\phi_1)\dot\phi_1^2 \sin(\theta_1) L_0 \cos(\phi_0)\cos(\theta_0) \\
& +2m_1 L_1 \sin(\phi_1)\dot\phi_1 \cos(\theta_1)\dot\theta_1 L_0 \cos(\phi_0)\cos(\theta_0) - 2m_1 \dot L_1 \cos(\phi_1)\cos(\theta_1)\dot\theta_1 L_0 \cos(\phi_0)\cos(\theta_0) \\
& +m_1 L_1 \cos(\phi_1)\sin(\theta_1)\dot\theta_1^2 L_0 \cos(\phi_0)\cos(\theta_0) - \rho_2 L_0^2 \left(\cos(\phi_0)\right)^2 \dot\theta_0 \dot L_2 \\
& -2m_1 \dot L_1 \sin(\phi_1)\dot\phi_1 \cos(\theta_1) L_0 \cos(\phi_0)\sin(\theta_0) - m_1 L_1 \cos(\phi_1)\dot\phi_1^2 \cos(\theta_1) L_0 \cos(\phi_0)\sin(\theta_0) \\
& +m_1 L_0 \cos(\phi_0)\sin(\theta_0) + m_2 L_0 \cos(\phi_0)\sin(\theta_0) + Q_1 + 2m_2 L_2 \sin(\phi_2)\dot\phi_2 \sin(\theta_2)\dot\theta_2 L_0 \cos(\phi_0)\sin(\theta_0) \\
& +2m_2 L_0^2 \cos(\phi_0)\dot\theta_0 \sin(\phi_0)\dot\phi_0 - 2m_2 L_0 \left(\cos(\phi_0)\right)^2 \dot\theta_0 \dot L_0 - 2m_2 \dot L_2 \cos(\phi_2)\sin(\theta_2)\dot\theta_2 L_0 \cos(\phi_0)\sin(\theta_0) \\
& -m_2 L_2 \cos(\phi_2)\cos(\theta_2)\dot\theta_2^2 L_0 \cos(\phi_0)\sin(\theta_0) + 2m_2 \dot L_2 \sin(\phi_2)\dot\phi_2 \sin(\theta_2) L_0 \cos(\phi_0)\cos(\theta_0) \\
& -2m_2 \dot L_2 \cos(\phi_2)\cos(\theta_2)\dot\theta_2 L_0 \cos(\phi_0)\cos(\theta_0) - 2m_2 \dot L_2 \sin(\phi_2)\dot\phi_2 \cos(\theta_2) L_0 \cos(\phi_0)\sin(\theta_0) \\
& -m_2 L_2 \cos(\phi_2)\dot\phi_2^2 \cos(\theta_2) L_0 \cos(\phi_0)\sin(\theta_0) + m_2 L_2 \cos(\phi_2)\sin(\theta_2)\dot\theta_2^2 L_0 \cos(\phi_0)\cos(\theta_0) \\
& +\rho_2 g L_0 \cos(\phi_0)\sin(\theta_0) L_2 + 1/2\rho_0 g \cos(\phi_0)\sin(\theta_0) L_0^2 - 1/2\rho_2 L_2^2 \cos(\phi_2)\dot\phi_2^2 \cos(\theta_2) L_0 \cos(\phi_0)\sin(\theta_0) \\
& +2m_1 \dot L_1 \sin(\phi_1)\dot\phi_1 \sin(\theta_1) L_0 \cos(\phi_0)\cos(\theta_0) + 1/2\rho_2 L_2^2 \cos(\phi_2)\dot\phi_2^2 \sin(\theta_2) L_0 \cos(\phi_0)\cos(\theta_0) \\
& +1/2\rho_2 L_2^2 \cos(\phi_2)\sin(\theta_2)\dot\theta_2^2 L_0 \cos(\phi_0)\cos(\theta_0) - 2\rho_2 L_0 \left(\cos(\phi_0)\right)^2 \dot\theta_0 L_2 \dot L_0 - \rho_0 \left(\cos(\phi_0)\right)^2 \dot\theta_0 L_0^2 \dot L_0 \\
& +2m_1 L_0^2 \cos(\phi_0)\dot\theta_0 \sin(\phi_0)\dot\phi_0 + m_2 L_2 \cos(\phi_2)\dot\phi_2^2 \sin(\theta_2) L_0 \cos(\phi_0)\cos(\theta_0) \\
& -1/2\rho_1 L_1^2 \cos(\phi_1)\cos(\theta_1)\dot\theta_1^2 L_0 \cos(\phi_0)\sin(\theta_0) + 1/2\rho_1 L_1^2 \cos(\phi_1)\sin(\theta_1)\dot\theta_1^2 L_0 \cos(\phi_0)\cos(\theta_0) \\
& -2m_1 \dot L_1 \cos(\phi_1)\sin(\theta_1)\dot\theta_1 L_0 \cos(\phi_0)\sin(\theta_0) - 2\rho_1 \dot L_1 \sin(\phi_1)\dot\phi_1 \cos(\theta_1) L_0 \cos(\phi_0)\sin(\theta_0) L_1 \\
& -2\rho_1 \dot L_1 \cos(\phi_1)\cos(\theta_1)\dot\theta_1 L_0 \cos(\phi_0)\cos(\theta_0) L_1 + 2\rho_1 \dot L_1 \sin(\phi_1)\dot\phi_1 \sin(\theta_1) L_0 \cos(\phi_0)\cos(\theta_0) L_1 \\
& -\rho_1 L_0^2 \left(\cos(\phi_0)\right)^2 \dot\theta_0 \dot L_1 - \rho_1 \ddot L_1 \cos(\phi_1)\sin(\theta_1) L_0 \cos(\phi_0)\cos(\theta_0) L_1 + \rho_1 \ddot L_1 \cos(\phi_1)\cos(\theta_1) L_0 \cos(\phi_0)\sin(\theta_0) L_1 \\
& -\rho_2 \ddot L_2 \cos(\phi_2)\sin(\theta_2) L_0 \cos(\phi_0)\cos(\theta_0) L_2 + \rho_2 \ddot L_2 \cos(\phi_2)\cos(\theta_2) L_0 \cos(\phi_0)\sin(\theta_0) L_2 \\
& +m_1 \ddot L_1 \cos(\phi_1)\cos(\theta_1) L_0 \cos(\phi_0)\sin(\theta_0) - m_1 \ddot L_1 \cos(\phi_1)\sin(\theta_1) L_0 \cos(\phi_0)\cos(\theta_0) \\
& +m_2 \ddot L_2 \cos(\phi_2)\cos(\theta_2) L_0 \cos(\phi_0)\sin(\theta_0) - m_2 \ddot L_2 \cos(\phi_2)\sin(\theta_2) L_0 \cos(\phi_0)\cos(\theta_0) \\
& +2/3\rho_0 \cos(\phi_0)\dot\theta_0 L_0^3 \sin(\phi_0)\dot\phi_0 + 1/2\rho_1 L_1^2 \cos(\phi_1)\dot\phi_1^2 \sin(\theta_1) L_0 \cos(\phi_0)\cos(\theta_0) \\
& -m_1 L_1 \cos(\phi_1)\cos(\theta_1)\dot\theta_1^2 L_0 \cos(\phi_0)\sin(\theta_0) + \rho_1 g L_0 \cos(\phi_0)\sin(\theta_0) L_1
\end{aligned} \tag{58}
$$

Spherical Coordinates

... page 3 & 4 ...

Spherical Coordinates

... pages 5 & 6 ...

Spherical Coordinates

...page 7.

$$
\begin{aligned}
b_8 = &-2\,m_2 L_2 \dot{\phi}_2 \dot{L}_2 - 2\,m_2 L_2 \sin(\phi_2)\sin(\theta_2) L_0 \sin(\phi_0)\dot{\phi}_0 \cos(\theta_0)\dot{\theta}_0 \\
&-2\,m_2 L_2 \sin(\phi_2)\sin(\theta_2)\dot{L}_0 \sin(\phi_0)\dot{\phi}_0 \sin(\theta_0) - m_2 L_2^{\,2} \cos(\phi_2)\dot{\theta}_2^{\,2} \sin(\phi_2) \\
&-2m_2 L_2 \cos(\phi_2)\dot{L}_0 \cos(\phi_0)\dot{\phi}_0 - m_2 L_2 \sin(\phi_2)\sin(\theta_2) L_0 \cos(\phi_0)\dot{\phi}_0^{\,2} \sin(\theta_0) \\
&-m_2 L_2 \cos(\phi_2)\ddot{L}_0 \sin(\phi_0) + 1/2\,\rho_2 g \sin(\phi_2)\cos(\theta_2) L_2^{\,2} + 2\,m_2 L_2 \sin(\phi_2)\sin(\theta_2)\dot{L}_0 \cos(\phi_0)\cos(\theta_0)\dot{\theta}_0 \\
&+m_2 L_2 \cos(\phi_2) L_0 \sin(\phi_0)\dot{\phi}_0^{\,2} - 1/2\rho_2 L_2^{\,2} \cos(\phi_2)\ddot{L}_0 \sin(\phi_0) - m_2 L_2 \sin(\phi_2)\sin(\theta_2) L_0 \cos(\phi_0)\sin(\theta_0)\dot{\theta}_0^{\,2} \\
&-m_2 L_2 \sin(\phi_2)\cos(\theta_2) L_0 \cos(\phi_0)\dot{\phi}_0^{\,2} \cos(\theta_0) + m_2 L_2 \sin(\phi_2)\cos(\theta_2)\ddot{L}_0 \cos(\phi_0)\cos(\theta_0) \\
&+2m_2 L_2 \sin(\phi_2)\cos(\theta_2) L_0 \sin(\phi_0)\dot{\phi}_0 \sin(\theta_0)\dot{\theta}_0 + 1/2\,\rho_2 L_2^{\,2} \sin(\phi_2)\cos(\theta_2)\ddot{L}_0 \cos(\phi_0)\cos(\theta_0) \\
&-m_2 L_2 \sin(\phi_2)\cos(\theta_2) L_0 \cos(\phi_0)\cos(\theta_0)\dot{\theta}_0^{\,2} + \rho_2 L_2^{\,2} \sin(\phi_2)\cos(\theta_2) L_0 \sin(\phi_0)\dot{\phi}_0 \sin(\theta_0)\dot{\theta}_0 \\
&+1/2\rho_2 L_2^{\,2} \sin(\phi_2)\sin(\theta_2)\ddot{L}_0 \cos(\phi_0)\sin(\theta_0) - 1/2\,\rho_2 L_2^{\,2} \sin(\phi_2)\cos(\theta_2) L_0 \cos(\phi_0)\cos(\theta_0)\dot{\theta}_0^{\,2} \\
&-1/2\,\rho_2 L_2^{\,2} \sin(\phi_2)\sin(\theta_2) L_0 \cos(\phi_0)\dot{\phi}_0^{\,2} \sin(\theta_0) - 1/3\rho_2 \cos(\phi_2)\dot{\theta}_2^{\,2} \sin(\phi_2) L_2^{\,3} \\
&-1/2\,\rho_2 L_2^{\,2} \sin(\phi_2)\cos(\theta_2) L_0 \cos(\phi_0)\dot{\phi}_0^{\,2} \cos(\theta_0) + 1/2\rho_2 L_2^{\,2} \cos(\phi_2) L_0 \sin(\phi_0)\dot{\phi}_0^{\,2} \\
&-\rho_2 L_2^{\,2} \sin(\phi_2)\cos(\theta_2)\dot{L}_0 \cos(\phi_0)\sin(\theta_0)\dot{\theta}_0 - \rho_2 L_2^{\,2} \sin(\phi_2)\cos(\theta_2)\dot{L}_0 \sin(\phi_0)\dot{\phi}_0 \cos(\theta_0) \\
&+m_2 L_2 \sin(\phi_2)\sin(\theta_2)\ddot{L}_0 \cos(\phi_0)\sin(\theta_0) - \rho_2 \dot{\phi}_2 L_2^{\,2} \dot{L}_2 + m_2 L_2 \sin(\phi_2)\cos(\theta_2) \\
&-1/2\rho_2 L_2^{\,2} \sin(\phi_2)\sin(\theta_2) L_0 \cos(\phi_0)\sin(\theta_0)\dot{\theta}_0^{\,2} - \rho_2 L_2^{\,2} \sin(\phi_2)\sin(\theta_2) L_0 \sin(\phi_0)\dot{\phi}_0 \cos(\theta_0)\dot{\theta}_0 \\
&-\rho_2 L_2^{\,2} \cos(\phi_2)\dot{L}_0 \cos(\phi_0)\dot{\phi}_0 - \rho_2 L_2^{\,2} \sin(\phi_2)\sin(\theta_2)\dot{L}_0 \sin(\phi_0)\dot{\phi}_0 \sin(\theta_0) \\
&+\rho_2 L_2^{\,2} \sin(\phi_2)\sin(\theta_2)\dot{L}_0 \cos(\phi_0)\cos(\theta_0)\dot{\theta}_0 - 2m_2 L_2 \sin(\phi_2)\cos(\theta_2)\dot{L}_0 \sin(\phi_0)\dot{\phi}_0 \cos(\theta_0) \\
&-2\,m_2 L_2 \sin(\phi_2)\cos(\theta_2)\dot{L}_0 \cos(\phi_0)\sin(\theta_0)\dot{\theta}_0 + Q_8
\end{aligned}
\tag{65}
$$

$$
\begin{aligned}
b_9 = &-\rho_2 \ddot{L}_2 L_2 + 1/2\,\rho_2 L_2^{\,2} \dot{\phi}_2^{\,2} + 1/2\,\rho_2 L_0^{\,2} \dot{\phi}_0^{\,2} - m_2 \cos(\phi_2)\cos(\theta_2) \\
&+m_2 L_2 \dot{\phi}_2^{\,2} - \rho_2 \cos(\phi_2)\cos(\theta_2)\ddot{L}_0 \cos(\phi_0)\cos(\theta_0) L_2 + m_2 \cos(\phi_2)\cos(\theta_2) L_0 \cos(\phi_0)\cos(\theta_0)\dot{\theta}_0^{\,2} \\
&+m_2 \cos(\phi_2)\cos(\theta_2) L_0 \cos(\phi_0)\dot{\phi}_0^{\,2} \cos(\theta_0) + 2m_2 \cos(\phi_2)\cos(\theta_2)\dot{L}_0 \cos(\phi_0)\sin(\theta_0)\dot{\theta}_0 \\
&+2\,m_2 \cos(\phi_2)\cos(\theta_2)\dot{L}_0 \sin(\phi_0)\dot{\phi}_0 \cos(\theta_0) + m_2 \cos(\phi_2)\sin(\theta_2) L_0 \cos(\phi_0)\sin(\theta_0)\dot{\theta}_0^{\,2} \\
&-2\,m_2 \cos(\phi_2)\sin(\theta_2)\dot{L}_0 \cos(\phi_0)\cos(\theta_0)\dot{\theta}_0 + m_2 \cos(\phi_2)\sin(\theta_2) L_0 \cos(\phi_0)\dot{\phi}_0^{\,2} \sin(\theta_0) \\
&+2m_2 \cos(\phi_2)\sin(\theta_2)\dot{L}_0 \sin(\phi_0)\dot{\phi}_0 \sin(\theta_0) - \rho_2 \cos(\phi_2)\sin(\theta_2)\ddot{L}_0 \cos(\phi_0)\sin(\theta_0) L_2 \\
&-2\,\rho_2 \cos(\phi_2)\sin(\theta_2)\dot{L}_0 \cos(\phi_0)\cos(\theta_0)\dot{\theta}_0 L_2 + 2\,\rho_2 \cos(\phi_2)\sin(\theta_2)\dot{L}_0 \sin(\phi_0)\dot{\phi}_0 \sin(\theta_0) L_2 \\
&+\rho_2 \sin(\phi_2) L_0 \sin(\phi_0)\dot{\phi}_0^{\,2} L_2 - 2\rho_2 \sin(\phi_2)\dot{L}_0 \cos(\phi_0)\dot{\phi}_0 L_2 \\
&-2\,m_2 \cos(\phi_2)\cos(\theta_2) L_0 \sin(\phi_0)\dot{\phi}_0 \sin(\theta_0)\dot{\theta}_0 - m_2 \cos(\phi_2)\cos(\theta_2)\ddot{L}_0 \cos(\phi_0)\cos(\theta_0) \\
&+2m_2 \cos(\phi_2)\sin(\theta_2) L_0 \sin(\phi_0)\dot{\phi}_0 \cos(\theta_0)\dot{\theta}_0 - m_2 \cos(\phi_2)\sin(\theta_2)\ddot{L}_0 \cos(\phi_0)\sin(\theta_0) \\
&+\rho_2 \cos(\phi_2)\cos(\theta_2) L_0 \cos(\phi_0)\dot{\phi}_0^{\,2} \cos(\theta_0) L_2 + 2\,\rho_2 \cos(\phi_2)\cos(\theta_2)\dot{L}_0 \cos(\phi_0)\sin(\theta_0)\dot{\theta}_0 L_2 \\
&+2\rho_2 \cos(\phi_2)\cos(\theta_2)\dot{L}_0 \sin(\phi_0)\dot{\phi}_0 \cos(\theta_0) L_2 + \rho_2 \cos(\phi_2)\sin(\theta_2) L_0 \cos(\phi_0)\dot{\phi}_0^{\,2} \sin(\theta_0) L_2 \\
&+\rho_2 \cos(\phi_2)\cos(\theta_2) L_0 \cos(\phi_0)\cos(\theta_0)\dot{\theta}_0^{\,2} L_2 + \rho_2 \cos(\phi_2)\sin(\theta_2) L_0 \cos(\phi_0)\sin(\theta_0)\dot{\theta}_0^{\,2} L_2 \\
&-2\rho_2 \cos(\phi_2)\cos(\theta_2) L_0 \sin(\phi_0)\dot{\phi}_0 \sin(\theta_0)\dot{\theta}_0 L_2 + 1/2\,\rho_2 L_0^{\,2} \left(\cos(\phi_0)\right)^2 \dot{\theta}_0^{\,2} + m_2 L_2 \left(\cos(\phi_2)\right)^2 \dot{\theta}_2^{\,2} \\
&+1/2\rho_2 L_2^{\,2} \left(\cos(\phi_2)\right)^2 \dot{\theta}_2^{\,2} + 2\,\rho_2 \cos(\phi_2)\sin(\theta_2) L_0 \sin(\phi_0)\dot{\phi}_0 \cos(\theta_0)\dot{\theta}_0 L_2 - m_2 \sin(\phi_2)\ddot{L}_0 \sin(\phi_0) \\
&-m_2 \ddot{L}_2 + Q_9 - \rho_2 g L_0 \cos(\phi_0)\cos(\theta_0) - \rho_2 g L_2 \cos(\phi_2)\cos(\theta_2) - \rho_2 \sin(\phi_2)\ddot{L}_0 \sin(\phi_0) L_2 \\
&-2m_2 \sin(\phi_2)\dot{L}_0 \cos(\phi_0)\dot{\phi}_0 + m_2 \sin(\phi_2) L_0 \sin(\phi_0)\dot{\phi}_0^{\,2} - 1/2\,\rho_2 \dot{L}_2^{\,2} + 1/2\,\rho_2 \dot{L}_0^{\,2}
\end{aligned}
\tag{66}
$$

The above equations of motion are implemented in MATLAB and solved as a function of time by expressing them in state-space form.

C. External Forces

1. Aerodynamic Kite Forces

The major external forces acting on the kite-tow system that are not modeled thus far in the equations of motion are the lift and drag forces from the kite, together with the drag forces on the tether. The kite is assumed to be controlled by manipulating its angle of attack and roll angle. Thus, in this study, its attitude dynamics are ignored. The lift and drag forces due to the kite are derived using a velocity coordinate system, as shown in 4.

Cartesian Coordinates

$$\begin{bmatrix} M & C \\ C^T & 0 \end{bmatrix} \begin{bmatrix} \ddot{\vec{r}} \\ \dot{\vec{\omega}} \\ \nu \end{bmatrix} = F, \qquad \ddot{\vec{r}} = \begin{bmatrix} \ddot{\vec{r}}_0 \\ \ddot{\vec{r}}_1 \\ \ddot{\vec{r}}_2 \end{bmatrix}, \quad \vec{\omega} = \begin{bmatrix} \dot{\vec{\omega}}_1 \\ \dot{\vec{\omega}}_2 \end{bmatrix}, \quad \vec{\nu} = \begin{bmatrix} \nu_0 \\ \nu_1 \\ \nu_2 \end{bmatrix}$$

$$M = \begin{bmatrix} \sum_{k=0}^{3} \xi_k & \frac{1}{2}\xi_1 & \frac{1}{2}\xi_2 & 0 & 0 \\ \frac{1}{2}\xi_1 & \xi_1 + m_1 I_3 & 0 & 0 & 0 \\ \frac{1}{2}\xi_2 & 0 & \xi_2 + m_2 I_3 & 0 & 0 \\ 0 & 0 & 0 & J_1 & 0 \\ 0 & 0 & 0 & 0 & J_2 \end{bmatrix}, \quad C = \begin{bmatrix} \vec{r}_0 & \nabla_{\vec{r}_0} c_1 & \nabla_{\vec{r}_0} c_2 \\ 0 & \nabla_{\vec{r}_1} c_1 & 0 \\ 0 & 0 & \nabla_{\vec{r}_2} c_2 \\ 0 & 2P_{R_1}\left(\nabla_{R_1} c_1\right) & 0 \\ 0 & 0 & 2P_{R_2}\left(\nabla_{R_2} c_2\right) \end{bmatrix},$$

$$F = \begin{bmatrix} \vec{F}_0 - \frac{1}{2} g \mu_0 l_0 \vec{1}_3 - \sum_{k=1}^{2} \frac{1}{2} g \mu_k l_k \vec{1}_3 \\ \vec{F}_1 - \frac{1}{2} g \mu_1 l_1 \vec{1}_3 - g m_1 \vec{1}_3 \\ \vec{F}_2 - \frac{1}{2} g \mu_2 l_2 \vec{1}_3 - g m_2 \vec{1}_3 \\ \vec{M}_1 - \omega_1 \times J_1 \omega_1 \\ \vec{M}_2 - \omega_2 \times J_2 \omega_2 \\ \nabla_{\vec{r}_0} \dot{c}_0^T \dot{\vec{r}}_0 \\ -\nabla_{\vec{r}_0} \dot{c}_1^T \dot{\vec{r}}_0 - \nabla_{\vec{r}_1} \dot{c}_1^T \dot{\vec{r}}_1 - 2P_{R_1}\left(\nabla_{R_1} \dot{c}_1\right)^T \omega_1 \\ -\nabla_{\vec{r}_0} \dot{c}_2^T \dot{\vec{r}}_0 - \nabla_{\vec{r}_2} \dot{c}_2^T \dot{\vec{r}}_2 - 2P_{R_2}\left(\nabla_{R_2} \dot{c}_2\right)^T \omega_2 \end{bmatrix}$$

$$\dot{c}_0 = \vec{r}_0^T \dot{\vec{r}}_0, \qquad \dot{c}_k = \left(\vec{r}_k + R_k \vec{r}_{\mathrm{T}} - \vec{r}_0\right)^T \left(\dot{\vec{r}}_k + R_k \omega_k \times \vec{r}_{\mathrm{T}} - \dot{\vec{r}}_0\right)$$

$$\nabla_{\vec{r}_0} c_0 = \vec{r}_0, \quad -\nabla_{\vec{r}_0} \dot{c}_k = \nabla_{\vec{r}_k} \dot{c}_k = \left(\dot{\vec{r}}_k + R_k \omega_k \times \vec{r}_{\mathrm{T}} - \dot{\vec{r}}_0\right),$$

$$2P_{R_k}\left(\nabla_{R_k} c_k\right) = \vec{r}_{\mathrm{T}} \times R_k^T \left(\vec{r}_k - \vec{r}_0\right), \quad -\nabla_{\vec{r}_0} c_k = \nabla_{\vec{r}_k} c_k = \vec{r}_k + R\vec{r}_{\mathrm{T}} - \vec{r}_0,$$

$$2P_{R_k}\left(\nabla_{R_k} \dot{c}_k\right) = R^T \left(\vec{r} - \vec{r}_0\right) \times \left(\omega \times \vec{r}_{\mathrm{T}}\right) + R^T \left(\dot{\vec{r}} - \dot{\vec{r}}_0\right) \times \vec{r}_{\mathrm{T}},$$

Dual Airfoils

Airfoil model: index-1 DAE

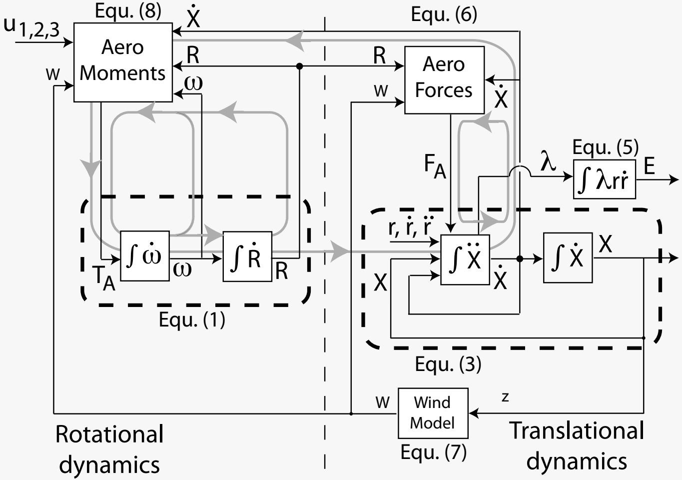
- 50 differential states
- 3 algebraic states
- 8 controls

Wind turbulence model:

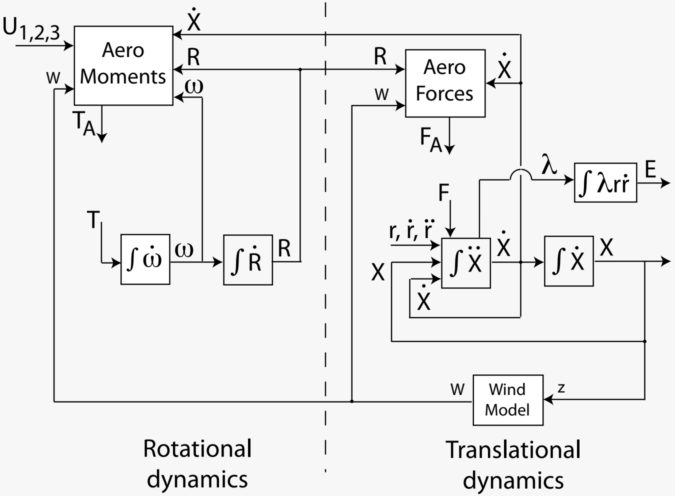
$$\dot{w}_{\diamond\star} = -\frac{w_{\diamond\star}}{\tau} + u_{\diamond\star}, \quad \diamond \in \{x, y, z\}, \star \in \{1, 2\}$$

- 6 differential states
- 6 controls

Offline Trajectory Optimization



Offline Trajectory Optimization



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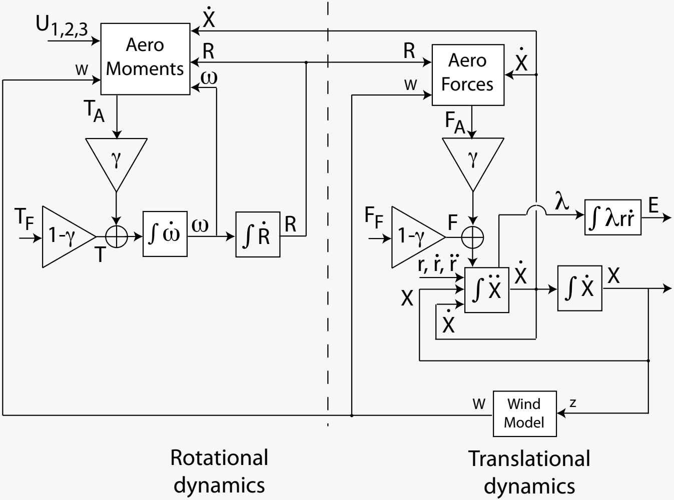

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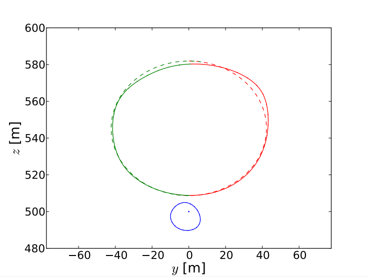

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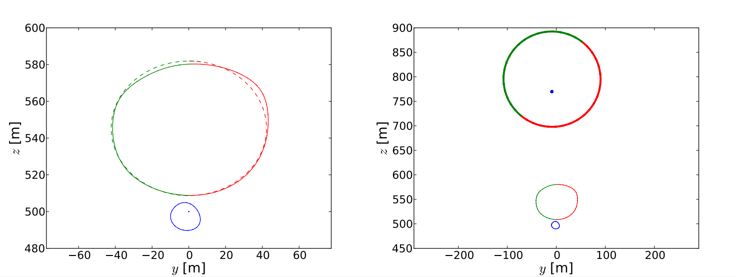

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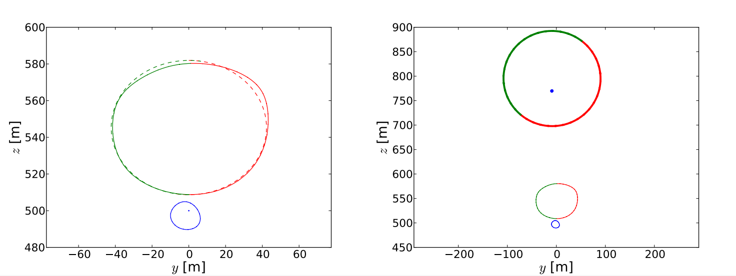

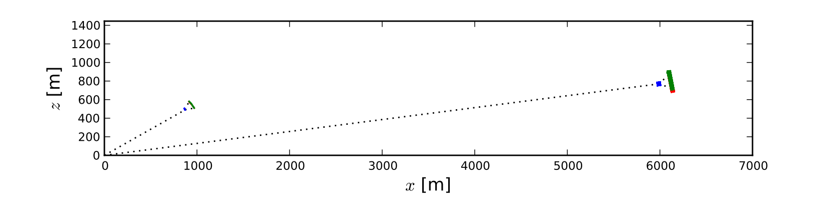

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